Topics

- Tate-module arithmetic when $k \neq \bar{k}$.

Lect 3, page 9.

Prop $X/\operatorname{Spec} k$ smooth connected proper curve.

$f : X \longrightarrow \mathbb{P}^1_k$ merom fct.

f isomorphism $\Leftrightarrow$ $\deg f = 1$.

Proof f finite locally free.

$\longmapsto$ $X \simeq \underline{\operatorname{Spec}}\, f_* \mathcal{O}_X$.

+ f induced from the $\mathcal{O}_{\mathbb{P}^1_k}$ -algebra

structure $\mathcal{O}_{\mathbb{P}^1_k} \longrightarrow f_* \mathcal{O}_X$.

This map is an isom $\Leftrightarrow$ $f_* \mathcal{O}_X$ of rank 1 as $\mathcal{O}_{\mathbb{P}^1_k}$-module. $\square$

$\mathbb{P}^1_k(\mathbb{C})$ & $E(\mathbb{C})$ cannot be isomorphic since genus is 0 & 1 in the cases.

Lect. 2, page 7

Question 2) $\Rightarrow$ 3) Prop:

$U \subseteq V(I) \overset{i}{\hookrightarrow} \mathbb{A}^n_k$. Then

$U_{\bar{k}}$ is regular $\Rightarrow$ the following ex & loc. split

$$0 \to I/I^2 \to i^*\Omega^1_{\mathbb{A}^n/k} \to \Omega^1_{U/k} \to 0$$

Question Why if $k \neq \bar{k}$, still

$$\begin{array}{ccc} QIsog_\ell(E) & \longleftrightarrow & \{\Lambda_\ell \subseteq V_\ell E\} \ ? \\ \cap\backslash & " & /\!/ \\ & QIsog_\ell(E_{\bar{k}}) & \end{array}$$

In general: $Gal(\bar{k}/k) \circlearrowright E[\ell^n](\bar{k})$

$$\begin{tikzcd} & \sigma \arrow[d, maps to] & x \arrow[d, maps to] \\ \mathrm{Spec}\,\bar{k} \arrow[r, "\sigma"] & \mathrm{Spec}\,\bar{k} \arrow[r, "x"] & E \end{tikzcd}$$

$\sigma \cdot x$

extends to an action $\mathrm{Gal}(\bar{k}/k)$

$\curvearrowright T_\ell E, V_\ell E$

$$\mathrm{QIsog}_\ell(E) \xrightarrow{\cong} \{\mathrm{Gal}(\bar{k}/k)\text{-stable lattices } \Lambda_\ell \subseteq V_\ell E\}$$

Boils down to:

$$\mathrm{Isog}(E) \hookrightarrow \mathrm{Isog}(E_{\bar{k}})$$
$$\ni (E', \phi)$$

Then $(E', \phi) \in LHS \iff \ker \phi$ is defined over k

$\iff \{(\ker \phi)(\bar{k}) \subseteq E[\ell^n](\bar{k})\}$ is $\mathrm{Gal}(\bar{k}/k)$-stable.

Tate Conjecture k is fin gen over $\mathbb{F}_p$ or $\mathbb{Q}$.

Then $\mathrm{Hom}(E, E')_\ell = \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, T_\ell E')^{\mathrm{Gal}(\bar{k}/k)}$

<u>Question</u> S G/S fin loc free

$G \curvearrowright X$

If $S' \to S$, what properties does

$$(G\backslash X)_{S'} \xleftarrow{\ ?\ } G_{S'}\backslash X_{S'}$$ have ?

$S' \to S$ flat $\Rightarrow$ isomorphism.

(<u>locally</u> $G\backslash X = \operatorname{Spec} A^G$

$$0 \to A^G \to A \xrightarrow{\mu^* - p^*} H \otimes_R A$$

Definition stable under flat base change.)

Not iso in general!

$G = \{\pm 1\} \curvearrowright \mathbb{A}^1_{\mathbb{Z}} = X$

Then $G\backslash X \cong \mathbb{A}^1_{\mathbb{Z}}$

but $$G_{\mathbb{F}_2}\backslash X_{\mathbb{F}_2} \longrightarrow (G\backslash X)_{\mathbb{F}_2}$$

$$\operatorname{Spec} \mathbb{F}_2[t] \longrightarrow \operatorname{Spec} \mathbb{F}_2[t]$$
$$t^2 \longleftarrow\!\shortmid \; t$$

Prop If $G \curvearrowright X$ freely, then base change map iso $\forall\, S' \to S$.

(see e.g. van-der-Geer — Moonen.)

Lect. 11 p5

Know $E[p]$ deg p^2.

$\Longrightarrow$ $E[p]_k \longrightarrow \operatorname{Spec} k$ still deg p^2.

$\wr$ artinian k-scheme.

$$\cong \coprod_{x \in E[p](k)} \operatorname{Spec} A_x$$

A_x local artinian k-algebra

$$p^2 = \sum_x \dim_k A_x$$

$$A_x = \mathcal{O}_{E[p]_k, x}.$$

$E[p](\bar{k})$ finite p-torsion group.

Counting arg from above:

$$|E[p](\bar{k})| \in \{1, p, p^2\}.$$

Case p^2 $\iff$ $\dim_{\bar{k}} A_x = 1 \ \forall x$

$\iff$ $A_x = \bar{k}$

$\iff$ $E[p]_{\bar{k}}$ reduced.

$\iff$ $[p] : E \to E$ unramified

This is not case since

$$[p]^* : \mathrm{Lie}\, E \to \mathrm{Lie}\, E$$

$$= \cdot p = 0$$

Lect 12 p. 17 (Recall $E^\vee = \mathrm{Pic}^0_{E/S}$)

$\phi : E_1 \to E_2$ get $\phi^\vee : E_2^\vee \to E_1^\vee$

$$\phi^* : E_2 \xrightarrow[\cong]{\lambda} E_2^\vee \xrightarrow{\phi^\vee} E_1^\vee \xrightarrow[\cong]{\lambda} E_1 .$$

(ϕ^* & $\phi^\vee$ differ in same way as

$f^\vee : W^\vee \longrightarrow V^\vee$ for $W, V / k$ - vsp

resp. $f^* : W \longrightarrow V$ $W^\vee, V^\vee$ dual spaces

& f^* respecting

non-deg pairings $V \times V \to k$
$W \times W \to k$.)

<u>Prop</u> Every elliptic curve has a unique <u>principal polarization</u>.

$\hookrightarrow$ $\lambda : E \xrightarrow{\simeq} E^\vee$ induced (over k)

from an ample line bundle $\mathcal{L}$ via

$$x \longmapsto t_x^* \mathcal{L} \otimes \mathcal{L}^\vee .$$

(In this sense λ's are canonical.)

<u>Sheet 6 Ex 3 b)</u> char $k = p$, E/k EC

$$\operatorname{End}(E) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \hookrightarrow \varprojlim_n \operatorname{End}(E[p^n])$$

$\alpha \in \mathrm{End}(E[p^{n+1}])$

get $\alpha E[p^n] \subseteq E[p^n]$ since on S-val points, $\alpha(S)$ preserves $E(S)[p^n]$.

$\Longrightarrow$ Well defined RHS.

Assume α on LHS $\mapsto 0$.

May assume $p^{-1}\alpha \notin \mathrm{End}(E)$.

If $\alpha \mapsto 0$, in particular $\alpha|_{E[p]} = 0$.

$$\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "{[p]}"'] & & E \\ & E \arrow[ur, "\beta"'] & \end{tikzcd}$$

$\exists$ factorization

$\Longrightarrow p \mid \alpha$ $\lightning$

<u>In general</u> $E[p^\infty] := \varinjlim_n E[p^n]$ <u>p-divisible group</u>.

formally behave like ℓ-adic Tate module but at p.

$X/_k$ smooth, proper. Then $H^0(X, \mathcal{O}_X)$ is étale k-alg.

<u>Reason</u> X smooth $\Rightarrow X_{\bar{k}}$ smooth

$\Rightarrow X_{\bar{k}}$ reduced $\Rightarrow H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}})$ is reduced artinian $\bar{k}$-alg and hence étale/$\bar{k}$

$(\cong \prod \bar{k})$

A smooth/k $\Leftrightarrow$ $A_{\bar{k}}$ smooth/$\bar{k}$

for A k-algebra.